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MINIMAL SUBMANIFOLDS WITH A PARALLEL OR A HARMONIC p-FORM

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2000 Mathematics Subject Classification: 53C42, 53C24.

Abstract

The purpose of this paper is to study the relations between the existence of minimal immersions of a Riemannian manifold M into another and some structural or topological properties of M. The properties on M which we consider are the existence of a parallel or a harmonic p-form.

1 Introduction

The purpose of this paper is to obtain some non existence results about minimal submanifolds. Let (M^m, g) be an m-dimensional Riemannian manifold isometrically immersed by ϕ in an n-dimensional Riemannian manifold (N^n, h) $(n > m)$. The Gauss equation allows us to obtain rigidity results in terms of geometry of (M^m, g) and (N^n, h). For example, as a first consequence of the Gauss equation, we get the following well known inequality in each point x of (M^m, g)

$$|H(x)|^2 \geq (1/m)((Scal(x)/m) - (m-1)\overline{K}^1(\phi(x))) \tag{1}$$

where $|H(x)|^2$ and $Scal(x)$ are respectively the square of the mean curvature of ϕ and the scalar curvature of (M^m, g) at x and $\overline{K}^1(\phi(x))$ is the largest sectional curvature of (N^n, h) at $\phi(x)$. In particular, if $\overline{K}^1$ has an upper bound and if $Scal > m(m-1)\overline{K}^1_{max}$ (where $\overline{K}^1_{max} = \max_N(\overline{K}^1)$) for at least a point of (M^m, g), there is no minimal immersion of (M^m, g) into (N^n, h).

Many other results were obtained, by assuming that (M^m, g) is endowed with some particular structures or topological properties (see for instance [1], [12] and [3]). First recall the results of Sampson ([12]) and Dajczer and Rodriguez ([3]). They proved that there is no minimal immersion of an m-dimensional Kaehlerian manifold $(m \geq 4)$ into a Riemannian manifold of negative constant sectional curvature. Later, El Soufi ([5]) obtained a generalization of this result by assuming a pinching of the sectional curvature of (N^n, h) and Hernandez ([9]) obtained the same conclusion under the negativity of the complex sectional curvature of (N^n, h). More recently, Petit and El Soufi ([6]) extend this result in the case where (M^m, g) is not necessarily Kaehlerian but has a parallel 2-form and where the isotropic curvature of (N^n, h) is negative (recall that the isotropic curvature of a Riemannian manifold is the restriction of the complex sectional curvature to isotropic tangent planes ([11])).

The section 1 of the present paper deals with some preliminaries. In the section 2, we consider the general case where (M^m, g) has a parallel p-form and we prove (theorem 3.1) that if (N^n, h) satisfies a curvature pinching condition, then there is no minimal immersion from (M^m, g) into (N^n, h). This is the generalization of the result of El Soufi stated in [5] for the case where (M^m, g) is Kaehlerian. Note that this theorem as well as the other results recalled above are of interest only if the sectional curvature of (N^n, h) is negative. However, in the theorem 3.2, we obtain the same conclusion with a new pinching condition for the case where (N^n, h) is not necessarily of negative sectional curvature but has a negative smallest sectional curvature. In the theorem 3.3, we study the particular case where (N^n, h) is the complex hyperbolic space $\mathbb{CH}^n(c)$ with constant holomorphic curvature equal to c and we prove that there is no totally real minimal immersion of a Riemannian manifold (M^m, g) with a parallel p-form into $\mathbb{CH}^n(c)$.

The compact manifolds with a parallel p-form are a particular case of manifolds with a harmonic p-form (or a nonzero p-th Betti number $b_p(M)$). In the section 3, we prove (theorem 4.1 and theorem 4.2) that for any compact manifold (M^m, g) with $b_p(M) \neq 0$ and isometrically immersed in a Riemannian manifold (N^n, h), there exists at least a point x of M so that

$$\frac{m}{\sqrt{p}}\left(\frac{p-1}{p}\right)|B(x)||H(x)| \geq k(x) - \left(\frac{p-1}{p}\right)((m-1)\overline{K}^1 + \overline{\rho}^1)(\phi(x))$$

and

$$m\left(\frac{p-1}{\sqrt{p}} + \frac{m-p-1}{\sqrt{m-p}}\right)|B(x)||H(x)| \geq Scal(x) - (m-2)((m-1)\overline{K}^1 + \overline{\rho}^1)(\phi(x))$$

where $|B(x)|$, $k(x)$ and $\overline{\rho}^1(\phi(x))$ denote respectively the norm of the second fundamental form of ϕ, the smallest eigenvalue of the Ricci curvature of (M^m, g) at x and the largest eigenvalue of the curvature operator of (N^n, h) at $\phi(x)$. El Soufi proved the first inequality for $p = 2$ in [5] and the second for $p = 2$ but only for $m = 4$. The first is optimal for the usual standard minimal embeddings of the Clifford torus and of the complex projective space in the sphere. These inequalities will be a consequence of a new lower bound of the curvature term in the Weitzenböck formula for p-forms (see the relation (5) and the propositions 4.1 and 4.2).

As a consequence of the previous inequalities, we deduce (corollary 4.2) that if (M^m, g) is minimally immersed in (N^n, h), if $\overline{\rho}^1$ is bounded above and if

$$\min_M(Scal) > (m-2)\left((m-1)\max_N(\overline{K}^1) + \max_N(\overline{\rho}^1)\right)$$

then (M^m, g) is a sphere of homology. This result can be viewed as a generalization of a theorem of Leung ([10]) which has shown that if a compact Riemannian manifold (M^m, g) is minimally immersed in a unit sphere and if the scalar curvature satisfies $Scal > m(m-2)$ then it is homeomorphic to an m-dimensional sphere.

2 Preliminaries and notations

Let (M^m, g) be an m-dimensional Riemannian manifold and let ϕ be an isometric immersion of (M^m, g) into an n-dimensional Riemannian manifold (N^n, h) $(n > m)$. The inner product and the norm induced by g and h on the tensors will be denoted respectively by $\langle\ ,\ \rangle$ and $|\ |^2$. Moreover, we denote respectively by R, ρ, Ric and $Scal$ the curvature tensor, the curvature operator, the Ricci tensor and the scalar curvature of (M^m, g) and by $\overline{R}$, $\overline{K}$, $\overline{\rho}$ and $\overline{Scal}$ the curvature tensor, the sectional curvature, the curvature operator and the scalar curvature of (N^n, h). We recall that for all vector field X, Y, Z, $W \in \Gamma(TN)$, $\overline{\rho}$ is defined by

$$\langle \overline{\rho}(X \wedge Y), Z \wedge W \rangle = \overline{R}(X, Y, Z, W)$$

Moreover, for all vector field X, $Y \in \Gamma(TM)$, the tensor $\overline{R}_\phi$ will be given by

$$\overline{R}_\phi(X, Y) = \sum_{i \leq m} \overline{R}(d\phi(X), d\phi(e_i), d\phi(Y), d\phi(e_i))$$

where $(e_i)_{1 \leq i \leq m}$ is an orthonormal frame on M. On the other hand, for all $x \in N$, we denote respectively, $\overline{K}^1(x)$ and $\overline{K}^0(x)$ the largest sectional curvature and the smallest sectional curvature at x and $\overline{\rho}^1(x)$ and $\overline{\rho}^0(x)$ the largest eigenvalue and the smallest eigenvalue of the curvature operator. Then, it is easy to see that

$$\overline{\rho}^0(x) \leq \overline{K}^0(x) \leq \overline{K}^1(x) \leq \overline{\rho}^1(x) \tag{2}$$

Now, let B be the second fundamental form of the immersion ϕ and let H be the mean curvature vector defined by: $H = (1/m) trace\, B$. The Gauss equation tells us that for any vector field X, Y, Z, $W \in \Gamma(TM)$, we have

$$R(X, Y, Z, W) = \overline{R}(X, Y, Z, W) + \langle B(X, Z), B(Y, W) \rangle - \langle B(X, W), B(Y, Z) \rangle \tag{3}$$

For the sake of completeness, we need now to recall briefly some definitions and properties about p-forms. Let $(e_i)_{1 \leq i \leq m}$ be a local orthonormal frame. Throughout this paper, for all q-tensor T, we will write $T_{i_1 \dots i_q}$ instead of $T(e_{i_1}, ..., e_{i_q})$ and then the inner product of two p-forms ω and θ of (M^m, g) will be

$$\langle \omega, \theta \rangle = \frac{1}{p!} \sum_{1 \leq i_1, \dots, i_p \leq m} \omega_{i_1 \dots i_p} \theta_{i_1 \dots i_p}$$

The inner product (or contraction) $i(X)\omega$ of a p-form ω with a vector field X on M is a $p-1$-form, defined by

$$(i(X))\omega(X_1, ..., X_{p-1}) = \omega(X, X_1, ..., X_{p-1}), \quad \forall\, X_1, ..., X_{p-1} \in \Gamma(TM)$$

More generally, if $X_1, ...X_q \in \Gamma(TM)$, then the inner product of the p-form ω with the q-tensor $X_1 \wedge ... \wedge X_q$ is the $p-q$-form defined by

$$(i(X_1 \wedge ... \wedge X_q)\omega)\,(Y_1, ..., Y_{p-q}) = \omega(X_q, ..., X_1, Y_1, ..., Y_{p-q}), \quad \forall\, Y_1, ..., Y_{p-q} \in \Gamma(TM)$$

Recall some elementary facts about inner and exterior products. Let ω and θ be respectively a p-form and a q-form and let X be a vector field on M, then

$$i(X)(\omega \wedge \theta) = i(X)\omega \wedge \theta + (-1)^p \omega \wedge i(X)\theta$$

and if $X^\star$ is the dual 1-form of the vector field X with respect to g, then $i(X)$ is in fact the adjoint of left exterior multiplication by $X^\star$, that is

$$\langle i(X)(\omega), \theta \rangle = \langle \omega, X^\star \wedge \theta \rangle$$

If M is orientable, we also need the following relation between the inner product and the Hodge operator $\star$ on (M^m, g) (see for instance [4])

$$i(X)(\star\omega) = (-1)^p \star (X^\star \wedge \omega)$$

On the other hand, if α is a 1-form which is real valued and β is a 1-form which is valued in a vector bundle, we define the 2-tensor $\alpha \vee \beta$ by

$$(\alpha \vee \beta)(X, Y) = \alpha(X)\beta(Y) + \alpha(Y)\beta(X) \tag{4}$$

We denote now by d, $d^\star$, ∇ and $\nabla^\star$ respectively the exterior differential and the codifferential acting on p-forms, the covariant derivative of (M^m, g) extended to p-forms and its adjoint with respect to g. The Hodge-de Rham Laplacian Δ acting on p-forms is given by

$$\Delta\omega = dd^\star\omega + d^\star d\omega$$

To compare this Laplacian to the "rough" Laplacian $\nabla^\star\nabla$, one has the Weitzenböck formula, reading as

$$\Delta\omega = \nabla^\star\nabla\omega + \mathcal{R}_p(\omega), \ \ \forall\, \omega \in \Lambda^p(M)$$

Here $\mathcal{R}_p \in End(\Lambda^p(M))$ is a bundle endomorphism, given by

$$\mathcal{R}_p(\omega)(X_1, ..., X_p) = \sum_{ij} (-1)^i [R(e_j, X_i)\omega](e_j, X_1, ..., \widehat{X_i}, ..., X_p), \ \ \forall X_1, ..., X_p \in \Gamma(TM)$$

where $(e_i)_{1 \leq i \leq m}$ is a local orthonormal frame and

$$R(X, Y)\omega = \nabla_{[X,Y]}\omega - [\nabla_X, \nabla_Y]\omega \ \ \forall\, X, Y \in \Gamma(TM)$$

An easy consequence of the Weitzenböck formula is the following

$$\frac{1}{2}\Delta|\omega|^2 = \langle \Delta\omega, \omega \rangle - |\nabla\omega|^2 - \langle \mathcal{R}_p(\omega), \omega \rangle \tag{5}$$

In the sequel, we need to explicit the expression of $\mathcal{R}_p$. A straightforward calculation gives us

$$\langle \mathcal{R}_p(\omega), \omega \rangle = \sum_{i,j} Ric_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle - \frac{1}{2} \sum_{i,j,k,l} R_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega \rangle \tag{6}$$

and the last term is zero when $p = 1$.

3 Geometry of submanifolds having a parallel p-form

The first result of this section is the following theorem

Theorem 3.1 *Let (M^m, g) be an m-dimensional Riemannian manifold admitting a non-trivial parallel p-form ($1 \leq p \leq m$) and let (N^n, h) be an n-dimensional Riemannian manifold ($n > m$). If for any $x \in (N^n, h)$, we have*

$$(m-1)\overline{K}^1(x) < (p-1)\overline{\rho}^0(x) \tag{7}$$

then, there is no minimal immersion from (M^m, g) into (N^n, h).

Remark 3.1:

1. For $p = 2$ and for even dimensional manifold (M^m, g), the pinching condition (7) can be refomulate as the negativity of the isotropic curvature ([6]). If (M^m, g) is Kaehlerian, this condition (7) is nothing but that obtained by El Soufi in [5] (theorem 2.2).

2. From the relation (2), we see that this theorem is of interest only if the sectional curvature of (N^n, h) is negative. For the hyperbolic space $\mathbb{H}^n$, the condition (7) is always satisfied for $p < m$ and then there is no minimal immersion of a manifold having a parallel p-form ($1 \leq p \leq m-1$) into $\mathbb{H}^n$. However, the embeddings of $\mathbb{H}^m$ in $\mathbb{H}^n$ ($m < n$) are totally geodesic, and taking the volume form of $\mathbb{H}^m$, we see that (7) is not satisfied for $p = m$.

In the following theorem, we obtain the same conclusion as in the theorem 3.1 with a new pinching condition where (N^n, h) is not necessarily of negative sectional curvature (in fact, there is no condition on $\overline{K}^1$).

Theorem 3.2 *Let (M^m, g) be an m-dimensional Riemannian manifold admitting a non-trivial parallel p-form ($1 \leq p \leq m$) and let (N^n, h) be an n-dimensional Riemannian manifold ($n > m$). If for any $x \in (N^n, h)$, we have*

$$\overline{Scal}(x) < (n-m)(n+m-1)\overline{K}^0(x) + (p(p-1)+(m-p)(m-p-1))\,\overline{\rho}^0(x) \tag{8}$$

then, there is no minimal immersion from (M^m, g) *into* (N^n, h)*.*

Remark 3.2: We will see in the proof that this theorem is of interest only if the smallest sectional curvature is negative that is $\overline{K}^0(x) < 0$ for all $x \in N$. For instance, let us consider the space $N^n = \mathbb{H}^r \times \mathbb{S}^s$ where $n = r + s$. Then $\overline{Scal} = -r(r-1) + s(s-1)$, $\overline{\rho}^1 = \overline{K}^1 = 1$ and $\overline{\rho}^0 = \overline{K}^0 = -1$. Now, let (M^m, g) be a Riemannian manifold of even dimension m and let $p = m/2$. Then we have $\overline{Scal} - (n-m)(n+m-1)\overline{K}^0 - (p(p-1)+(m-p)(m-p-1))\overline{\rho}^0 = 2(s^2 + rs - s - m^2/4)$ and it is easy to see that if r and m are great enough (for instance for a fixed s put $m = r$ great enough) then the condition (8) is satisfied and the conclusion of the theorem 3.2 holds for this example.

Proof of theorem 3.1: Let ϕ be a minimal immersion of (M^m, g) into (N^n, h) and assume that M has a nontrivial parallel p-form ω. Then $\langle \mathcal{R}_p(\omega), \omega\rangle = 0$ and from (6) we deduce

$$0 = \sum_{i,j} Ric_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle - \frac{1}{2}\sum_{i,j,k,l} R_{ijkl}\langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle \tag{9}$$

and the last term is zero for $p = 1$. Now from the Gauss formula (3), we obtain

$$\begin{aligned}\sum_{i,j} Ric_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle &= \sum_{i,j}(\overline{R}_\phi)_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle + m\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle \\ &- \sum_{i,j,k}\langle B_{ik}, B_{jk}\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle \end{aligned} \tag{10}$$

and

$$\begin{aligned}&\sum_{i,j,k,l} R_{ijkl}\langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle = \sum_{i,j,k,l}\overline{R}_{ijkl}\langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle \\ &\quad + \sum_{i,j,k,l}\langle B_{ik}, B_{jl}\rangle\langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle - \sum_{i,j,k,l}\langle B_{il}, B_{jk}\rangle\langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle \\ &\quad = \sum_{i,j,k,l}\overline{R}_{ijkl}\langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle + 2\sum_{i,j,k,l}\langle B_{ik}, B_{jl}\rangle\langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle\end{aligned} \tag{11}$$

by reporting (10) and (11) in (9), we get

$$
\begin{aligned}
0 &= \sum_{i,j}(\overline{R}_\phi)_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle + m\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle \\
&- \sum_{i,j,k}\langle B_{ik}, B_{jk}\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle - \frac{1}{2}\sum_{i,j,k,l}\overline{R}_{ijkl}\langle i(e_j\wedge e_i)\omega, i(e_l\wedge e_k)\omega\rangle \\
&- \sum_{i,j,k,l}\langle B_{ik}, B_{jl}\rangle\langle i(e_j\wedge e_i)\omega, i(e_l\wedge e_k)\omega\rangle \qquad (12)
\end{aligned}
$$

and the two last terms are zero for $p = 1$. Now, put $\mathcal{B}^+(\omega) = \sum_{i\leq m} i(e_i^\star)\omega \wedge B(e_i, .)$. The computation of the square of the norm of $\mathcal{B}^+(\omega)$ gives

$$
\begin{aligned}
|\mathcal{B}^+(\omega)|^2 &= \frac{1}{p!}\sum_{\substack{1\leq i_1,\dots,i_p\leq m\\ i,j}}\langle (i(e_i^\star)\omega\wedge B(e_i,.))_{i_1\dots i_p}, \left(i(e_j^\star)\omega\wedge B(e_j,.)\right)_{i_1\dots i_p}\rangle \\
&= \frac{1}{p!}\sum_{\substack{1\leq i_1,\dots,i_p\leq m\\ i,j,s,t}}(-1)^{s+t}\langle B_{ii_s}, B_{ji_t}\rangle\omega_{ii_1\dots\hat{i_s}\dots i_p}\omega_{ji_1\dots\hat{i_t}\dots i_p}
\end{aligned}
$$

where the indices with ˆ are omitted. Then

$$
\begin{aligned}
&|\mathcal{B}^+(\omega)|^2 \\
&= \frac{1}{p!}\sum_{\substack{1\leq i_1,\dots,i_p\leq m\\ i,j,s}}\langle B_{ii_s}, B_{ji_s}\rangle\omega_{ii_1\dots\hat{i_s}\dots i_p}\omega_{ji_1\dots\hat{i_s}\dots i_p} \\
&+ \frac{1}{p!}\sum_{\substack{1\leq i_1,\dots,i_p\leq m\\ i,j,s\neq t}}(-1)^{s+t}\langle B_{ii_s}, B_{ji_t}\rangle\omega_{ii_1\dots\hat{i_s}\dots i_p}\omega_{ji_1\dots\hat{i_t}\dots i_p} \\
&= \frac{1}{(p-1)!}\sum_{\substack{1\leq i_1,\dots,i_{p-1}\leq m\\ i,j,k}}\langle B_{ik}, B_{jk}\rangle\omega_{ii_1\dots i_{p-1}}\omega_{ji_1\dots i_{p-1}} \\
&- \frac{1}{p!}\sum_{\substack{1\leq i_1,\dots,i_p\leq m\\ i,j,s<t}}\langle B_{ii_s}, B_{ji_t}\rangle\omega_{ii_ti_1\dots\hat{i_s}\dots\hat{i_t}\dots i_p}\omega_{ji_si_1\dots\hat{i_s}\dots\hat{i_t}\dots i_p} \\
&- \frac{1}{p!}\sum_{\substack{1\leq i_1,\dots,i_p\leq m\\ i,j,s>t}}\langle B_{ii_s}, B_{ji_t}\rangle\omega_{ii_ti_1\dots\hat{i_t}\dots\hat{i_s}\dots i_p}\omega_{ji_si_1\dots\hat{i_t}\dots\hat{i_s}\dots i_p}
\end{aligned}
$$

$$= \sum_{i,j,k} \langle B_{ik}, B_{jk}\rangle \langle i(e_i)\omega, i(e_j)\omega\rangle - \frac{1}{(p-2)!} \sum_{\substack{1 \le i_1 \dots i_{p-2} \le m \\ i,j,k,l}} \langle B_{ik}, B_{jl}\rangle \omega_{ili_1\dots i_{p-2}} \omega_{jki_1\dots i_{p-2}}$$

$$= \sum_{i,j,k} \langle B_{ik}, B_{jk}\rangle \langle i(e_i)\omega, i(e_j)\omega\rangle + \frac{1}{(p-2)!} \sum_{\substack{1 \le i_1 \dots i_{p-2} \le m \\ i,j,k,l}} \langle B_{ik}, B_{jl}\rangle \omega_{iji_1\dots i_{p-2}} \omega_{kli_1\dots i_{p-2}}$$

$$= \sum_{i,j,k} \langle B_{ik}, B_{jk}\rangle \langle i(e_i)\omega, i(e_j)\omega\rangle + \sum_{i,j,k,l} \langle B_{ik}, B_{jl}\rangle \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle$$

Note that if M is Kaehlerian and ω is the Kaehler form then $|\mathcal{B}^+(\omega)|^2 = |B^+|^2$, where B^+ is the holomorphic part of B (i.e. $B^+(X,Y) = \frac{1}{2}(B(X,Y) + B(JX,JY))$ where $\omega(X,Y) = \langle JX, Y\rangle$).

Now, combining the above relation with (12), we obtain

$$|\mathcal{B}^+(\omega)|^2 = m \sum_{i,j} \langle B_{ij}, H\rangle \langle i(e_i)\omega, i(e_j)\omega\rangle + \sum_{i,j} (\overline{R}_\phi)_{ij} \langle i(e_i)\omega, i(e_j)\omega\rangle$$
$$- \frac{1}{2} \sum_{i,j,k,l} \overline{R}_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle \tag{13}$$

where the last term is zero if $p = 1$. Putting $X^{i_1\dots i_p} = \sum_{i \le m} \omega_{ii_1\dots i_{p-1}} e_i$ and $\theta^{i_1\dots i_{p-2}} = \frac{1}{2} \sum_{1 \le i,j \le m} \omega_{iji_1\dots i_{p-2}} e_i^\star \wedge e_j^\star$, we have

$$\sum_{i,j} (\overline{R}_\phi)_{ij} \langle i(e_i)\omega, i(e_j)\omega\rangle = \frac{1}{(p-1)!} \sum_{i_1,\dots,i_{p-1}} \overline{R}_\phi(X^{i_1\dots i_p}, X^{i_1\dots i_p})$$
$$\le \frac{(m-1)}{(p-1)!} \overline{K}^1(\phi(x)) \sum_{i_1,\dots,i_{p-1}} |X^{i_1\dots i_p}|^2 = p(m-1)\overline{K}^1(\phi(x))|\omega|^2 \tag{14}$$

and

$$\frac{1}{2} \sum_{i,j,k,l} \overline{R}_{ijkl} \langle i(e_j \wedge e_i)\omega, i(e_l \wedge e_k)\omega\rangle = \frac{2}{(p-2)!} \sum_{i_1,\dots,i_{p-2}} \overline{\rho}(\theta^{i_1\dots i_{p-2}}, \theta^{i_1\dots i_{p-2}})$$
$$\ge \frac{2}{(p-2)!} \overline{\rho}^0(\phi(x)) \sum_{i_1,\dots,i_{p-2}} |\theta^{i_1\dots i_{p-2}}|^2$$
$$= \frac{1}{(p-2)!} \overline{\rho}^0(\phi(x)) \sum_{i_1,\dots,i_p} {\omega^2}_{i_1\dots i_p} = p(p-1)\overline{\rho}^0(\phi(x))|\omega|^2 \tag{15}$$

consequently, for all $p \in \{1, ..., m\}$, it follows from (13) and the fact that $H = 0$

$$|\mathcal{B}^+(\omega)|^2(x) \leq p(m-1)\overline{K}^1(\phi(x))|\omega|^2 - p(p-1)\overline{\rho}^0(\phi(x))|\omega|^2$$

From this we conclude that if $(p-1)\overline{\rho}^0(x) > (m-1)\overline{K}^1(x)$ holds for any $x \in N$, then there is no minimal immersion from (M^m, g) into (N^n, h). □

Remark 3.3: In the theorem 3.1, the nonpositivity of the curvature operator of (N^n, h) is not required. In ([2]) Corlette proved (theorem 3.1) a similar result for harmonic maps ϕ from (M^m, g) to (N^n, h) (in fact he assumes that ϕ is a twisted harmonic map which is not necessary) by assuming only the nonpositivity of the curvature operator of (N^n, h) without pinching condition. Indeed, under this hypothesis he proved that if (M^m, g) is compact and has a parallel p-form then $\nabla^\star(d\phi \wedge \omega) = 0$ (here ∇ is the pullback of the Levi-Civita connection of TN). Note that Corlette doesn't obtain a result of non existence. Moreover if ϕ is a minimal isometric immersion, then ϕ is a harmonic map and the theorem of Corlette can be proved without the compacity of M and a short computation shows that $\nabla^\star(d\phi \wedge \omega) = -\mathcal{B}^+(\omega)$.

Proof of theorem 3.2: From the relation (13) and the inequality (15) of the previous proof, we deduce that if ϕ is a minimal immersion, we have

$$|\mathcal{B}^+(\omega)|^2 \leq \sum_{i,j} (\overline{R}_\phi)_{ij} \langle i(e_i)\omega, i(e_j)\omega\rangle - p(p-1)\overline{\rho}^0(\phi(x))|\omega|^2$$

Since M is locally oriented, we can define locally the Hodge operator $\star$ and the $m-p$-form $\star\omega$. But $|\mathcal{B}^+(\star\omega)|^2$ is independent on the choice of the orientability and is consequently globally defined. Then we have

$$\begin{aligned}|\mathcal{B}^+(\omega)|^2 + |\mathcal{B}^+(\star\omega)|^2 \leq & \sum_{i,j} (\overline{R}_\phi)_{ij} \left(\langle i(e_i)\omega, i(e_j)\omega\rangle + \langle i(e_i)\star\omega, i(e_j)\star\omega\rangle\right) \\ & - \left(p(p-1) + (m-p)(m-p-1)\right)\overline{\rho}^0(\phi(x))|\omega|^2 \qquad (16)\end{aligned}$$

Now using the properties about inner and exterior products recalled in the first section, we have

$$\begin{aligned}&\sum_{i,j} \left((\overline{R}_\phi)_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle + (\overline{R}_\phi)_{ij}\langle i(e_i)\star\omega, i(e_j)\star\omega\rangle\right) = \\ &\sum_{i,j} \left((\overline{R}_\phi)_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle + (\overline{R}_\phi)_{ij}\langle e_i^\star\wedge\omega, e_j^\star\wedge\omega\rangle\right) = \\ &\sum_{i,j} \left((\overline{R}_\phi)_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle + (\overline{R}_\phi)_{ij}\langle i(e_j)(e_i^\star\wedge\omega), \omega\rangle\right) =\end{aligned}$$

$$\sum_{i,j}(\overline{R}_\phi)_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle + \sum_i(\overline{R}_\phi)_{ii}|\omega|^2 - \sum_{i,j}(\overline{R}_\phi)_{ij}\langle e_i^\star \wedge i(e_j)\omega, \omega\rangle =$$
$$\sum_i(\overline{R}_\phi)_{ii}|\omega|^2 \tag{17}$$

A straightforward computation gives

$$\sum_i(\overline{R}_\phi)_{ii} \leq \overline{Scal}(\phi(x)) - (n-m)(n+m-1)\overline{K}^0(\phi(x))$$

and from (16) and (17) we deduce that for all $x \in M$

$$0 \leq \overline{Scal}(\phi(x)) - \Big((n-m)(n+m-1)\overline{K}^0 + (p(p-1)+(m-p)(m-p-1))\,\overline{\rho}^0\Big)(\phi(x))$$

Consequently if

$$\overline{Scal}(x) < (n-m)(n+m-1)\overline{K}^0(x) + (p(p-1)+(m-p)(m-p-1))\,\overline{\rho}^0(x)$$

for all $x \in N$, there is no minimal immersion from (M^m, g) into (N^n, h). Using the inequalities (2), we can easily see that the above condition implies

$$\left(\frac{m(m-1)}{p(p-1)+(m-p)(m-p-1)}\right)\overline{K}^0(x) < \overline{\rho}^0(x) \leq \overline{K}^0(x)$$

for all $x \in N$. And the theorem 3.2 is of interest only for $\overline{K}^0 < 0$. $\square$

To finish this section, we study the particular case where the ambient space is the complex hyperbolic space $\mathbb{CH}^n(c)$ with constant holomorphic curvature equal to c $(c < 0)$. Let us recall that in this case the curvature tensor of $\mathbb{CH}^n(c)$ has the expression

$$\overline{R}(X,Y,Z,W) = \frac{c}{4}\left(\langle X\wedge Y, Z\wedge W\rangle + \langle X\wedge Y, JZ\wedge JW\rangle + 2\langle X, JY\rangle\langle Z, JW\rangle\right) \tag{18}$$

for all X, Y, Z, $W \in \Gamma(T\mathbb{CH}^n(c))$. Here J denotes the complex structure of $\mathbb{CH}^n(c)$. For any isometric immersion ϕ of a Riemannian manifold (M^m, g) into $\mathbb{CH}^n(c)$, we define the $(1,1)$-tensor J_ϕ on M by $J_\phi X = \sum_{i\leq m}\langle Jd\phi(X), d\phi(e_i)\rangle e_i$, $\forall X \in \Gamma(TM)$ and for all orthonormal frame $(e_i)_{1\leq i\leq m}$. Recall that the immersion ϕ is said to be totally real if $J_\phi \equiv 0$.

For this kind of immersions we have the

Theorem 3.3 *Let (M^m, g) be an m-dimensional Riemannian manifold admitting a non-trivial parallel p-form $(p \geq 1)$. Then there is no minimal totally real immersion of (M^m, g) into $\mathbb{CH}^n(c)$.*

PROOF: Let ϕ be a minimal immersion of (M^m, g) into $\mathbb{CH}^n$ and assume that M has a non trivial parallel p-form ω. Then from (13) we have

$$|\mathcal{B}^+(\omega)|^2 = \sum_{i,j}(\overline{R}_\phi)_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle - \frac{1}{2}\sum_{i,j,k,l}\overline{R}_{ijkl}\langle i(e_j\wedge e_i)\omega, i(e_l\wedge e_k)\omega\rangle$$

Now the conclusion follows from a straightforward computation. Using (18) the above equality becomes

$$|\mathcal{B}^+(\omega)|^2 = \frac{c}{4}\Bigg(p(m-p)|\omega|^2 + 3\sum_{i,j\le m}\langle J_\phi e_i, J_\phi e_j\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle$$
$$- \sum_{i,j\le m}\langle i(e_j\wedge e_i)\omega, i(J_\phi e_j\wedge J_\phi e_i)\omega\rangle - \sum_{i,j\le m}\langle i(e_i\wedge J_\phi e_i)\omega, i(e_j\wedge J_\phi e_j)\omega\rangle\Bigg)$$

where $(e_i)_{1\le i\le m}$ is a local orthonormal frame. □

4 Geometry of submanifolds with $b_p(M) \neq 0$

Let (M^m, g) and (N^n, h) be two Riemannian manifolds and assume that (M^m, g) is compact. We use the same notations as in the previous sections. Moreover in this section, $k(x)$ denotes the smallest eigenvalue at x of the Ricci curvature of (M^m, g) and we put $k_0 = \min_M(k(x))$. On the other hand, if $\overline{K}^1$ is bounded above, we will set $\overline{K}^1_{max} = \max_N(\overline{K}^1)$ and $\overline{\rho}^1_{max} = \max_N(\overline{\rho}^1)$. The first result is the following theorem

Theorem 4.1 *Let (M^m, g) be a compact Riemannian manifold of dimension $m \ge 2$ so that $b_p(M) \neq 0$ for some $p \ge 1$. Then for any isometric immersion ϕ from (M^m, g) into an n-dimensional Riemannian manifold (N^n, h), there exists at least a point $x \in M$ so that*

$$\frac{m}{\sqrt{p}}\left(\frac{p-1}{p}\right)|B(x)||H(x)| \ge k(x) - \left(\frac{p-1}{p}\right)((m-1)\overline{K}^1 + \overline{\rho}^1)(\phi(x)) \qquad (19)$$

The following corollary is an immediate consequence of this theorem.

Corollary 4.1 *Let (M^m, g) be a compact Riemannian manifold of dimension $m \ge 2$ minimally immersed in an n-dimensional Riemannian manifold (N^n, h) $(n > m)$. If $\overline{\rho}^1$ is bounded above and if for an integer p so that $1 \le p \le m/2$, we have*

$$k_0 > \left(\frac{p-1}{p}\right)\left((m-1)\overline{K}^1_{max} + \overline{\rho}^1_{max}\right)$$

then $b_q(M) = 0$ *for* $q \in \{1, ..., p\}$.

Remark 4.1: The inequality (19) is an equality at each point for the standard embedding of $\mathbb{S}^p(\sqrt{p/m}) \times \mathbb{S}^{k_1}(\sqrt{k_1/m}) \times ... \times \mathbb{S}^{k_q}(\sqrt{k_q/m})$ into $\mathbb{S}^{m+r}$ where $p + k_1 + ... + k_q = m$ with $k_i \geq p$ $(1 \leq i \leq q)$. For the case $p = 2$, (19) is also an equality at each point for the standard embedding from $\mathbb{C}P^q$ with holomorphic curvature $2q/(q+1)$ into $\mathbb{S}^{q^2+2q}$.

This theorem 4.1 is a generalization of a result obtained by El Soufi (see theorem 3.1 of [5]) in the particular case $p = 2$.

To prove the theorem 4.1 we need the following proposition which gives an estimate of the term $\langle \mathcal{R}_p(\omega), \omega \rangle$ for any p-form ω.

Proposition 4.1 *Let* (M^m, g) *be an* m*-dimensional compact Riemannian manifold isometrically immersed in an* n*-dimensional Riemannian manifold* (N^n, h). *Then for any* $p \in \{1, ..., m\}$ *and for any* p*-form* ω *of* M*, we have for all* $x \in M$

$$\langle \mathcal{R}_p(\omega), \omega \rangle \geq p\left(pk(x) - (p-1)\left((m-1)\overline{K}^1 + \overline{\rho}^1\right)(\phi(x)) - m\left(\frac{p-1}{\sqrt{p}}\right)|H(x)||B(x)|\right)|\omega|^2$$

Before proving this proposition, we introduce the following p-tensor associated to any p-form ω of M and the isometric immersion ϕ

$$\mathcal{B}^-(\omega) = \frac{1}{(p-2)!}\sum_{i,i_1,...,i_{p-2} \leq m}\left(\left(i(e_i \wedge e_{i_1} \wedge ... \wedge e_{i_{p-2}})\omega\right) \vee B(e_i, .)\right) \otimes (e^\star_{i_1} \wedge ... \wedge e^\star_{i_{p-2}})$$

where $(e_i)_{1 \leq i \leq m}$ is an orthonormal frame at a point $x \in M$ and $\vee$ denotes the symmetric product defined in the preliminaries (see (4)). It will be convenient to choose the norm $|\mathcal{B}^-(\omega)|$ so that

$$|\mathcal{B}^-(\omega)|^2 = \frac{1}{(p-2)!}\sum_{jki_1...i_{p-2} \leq m}\left|\mathcal{B}^-(\omega)_{jki_1...i_{p-2}}\right|^2$$

Such a tensor has been introduced for the first time in [8] where the second fundamental form is replaced by the Hessian of a function. First note that if ω is a volume form at a point p of M where we have define the Hodge operator $\star$ so that $\omega = \star 1$, we deduce from (20) shown in the proof of the proposition 4.1, that

$$
\begin{aligned}
\frac{1}{2n}|\mathcal{B}^-(v_g)|^2 &= \frac{n-1}{n}\sum_{ijk\leq n}\langle B_{ik},B_{jk}\rangle\langle i(e_i)\star 1, i(e_j)\star 1\rangle \\
&-\frac{1}{n}\sum_{ijkl\leq n}\langle B_{ij},B_{kl}\rangle\langle i(e_l\wedge e_i)\star 1, i(e_k\wedge e_j)\star 1\rangle \\
&= \frac{n-1}{n}\sum_{ijk\leq n}\langle B_{ik},B_{jk}\rangle\langle e_i^\star, e_j^\star\rangle - \frac{1}{n}\sum_{ijkl\leq n}\langle B_{ij},B_{kl}\rangle\langle e_l^\star\wedge e_i^\star, e_k^\star\wedge e_j^\star\rangle \\
&= |B|^2 - n|H|^2 = |B-H\otimes g|^2
\end{aligned}
$$

In other words we obtain the square of the umbilicity tensor. Moreover if α denotes the Kaehler form of a Kaehlerian manifold, a straightforward calculation gives

$$|\mathcal{B}^-(\alpha)|^2 = 4|B^-|^2$$

where B^- is the anti-holomorphic part of B (i.e. $B^-(X,Y)=\frac{1}{2}(B(X,Y)-B(JX,JY))$ where $\alpha(X,Y)=\langle JX,Y\rangle$).

On the other hand we will see in the proof of theorem 4.1 that $\mathcal{B}^-(\omega)$ is vanishing identically for the standard embedding of $\mathbb{S}^p(\sqrt{p/m})\times\mathbb{S}^{k_1}(\sqrt{k_1/m})\times ... \times\mathbb{S}^{k_q}(\sqrt{k_q/m})$ into $\mathbb{S}^{m+r}$ where $p+k_1+...+k_q=m$ with $k_i\geq p$ $(1\leq i\leq q)$.

Proof of the proposition 4.1: For $p=1$ and from the relation (6), the equality of the proposition is obvious. Suppose now that $p\geq 2$. Let $x\in M$ and let $(e_i)_{1\leq i\leq m}$ be a local orthonormal frame in a neighborhood of x. We have

$$
\begin{aligned}
&\frac{(p-2)!}{2}|\mathcal{B}^-(\omega)|^2 = \frac{1}{2}\sum_{i,j,i_1,\dots,i_{p-2}}\left|\mathcal{B}^-(\omega)_{i,j,i_1,\dots,i_{p-2}}\right|^2 = \\
&= \frac{1}{2}\sum_{i,j,k,l,i_1,\dots,i_{p-2}}\langle\left(i(e_i\wedge e_{i_1}\wedge ...\wedge e_{i_{p-2}})\omega\vee B(e_i,.)\right)_{kl}, \left(i(e_j\wedge e_{i_1}\wedge ...\wedge e_{i_{p-2}})\omega\vee B(e_j,.)\right)_{kl}\rangle \\
&= \sum_{i,j,k,l,i_1,\dots,i_{p-2}}\left(i(e_i\wedge e_{i_1}\wedge ...\wedge e_{i_{p-2}})\omega\right)_k\left(i(e_j\wedge e_{i_1}\wedge ...\wedge e_{i_{p-2}})\omega\right)_k\langle B_{il},B_{jl}\rangle \\
&+ \sum_{i,j,k,l,i_1,\dots,i_{p-2}}\left(i(e_i\wedge e_{i_1}\wedge ...\wedge e_{i_{p-2}})\omega\right)_k\left(i(e_j\wedge e_{i_1}\wedge ...\wedge e_{i_{p-2}})\omega\right)_l\langle B_{il},B_{jk}\rangle \\
&= \sum_{i,j,k,l,i_1\dots i_{p-2}}\langle B_{il},B_{jl}\rangle\omega_{iki_1\dots i_{p-2}}\omega_{jki_1\dots i_{p-2}} + \sum_{i,j,k,l,i_1,\dots,i_{p-2}}\langle B_{il},B_{jk}\rangle\omega_{iki_1\dots i_{p-2}}\omega_{jli_1\dots i_{p-2}} \\
&= (p-1)!\sum_{i,j,k}\langle B_{ik},B_{jk}\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle - (p-2)!\sum_{i,j,k,l}\langle B_{ij},B_{kl}\rangle\langle i(e_l\wedge e_i)\omega, i(e_k\wedge e_j)\omega\rangle
\end{aligned}
$$

Finally, we have proved that

$$\frac{1}{2}|\mathcal{B}^-(\omega)|^2 = (p-1)\sum_{i,j,k}\langle B_{ik}, B_{jk}\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle - \sum_{i,j,k,l}\langle B_{ik}, B_{jl}\rangle\langle i(e_i\wedge e_j)\omega, i(e_k\wedge e_l)\omega\rangle \tag{20}$$

Now, combining this with the relations (10) and (11) and using the expression of $\langle \mathcal{R}_p(\omega), \omega\rangle$ (see (6)), we get

$$\begin{aligned}\frac{1}{2}|\mathcal{B}^-(\omega)|^2 &= \langle \mathcal{R}_p(\omega), \omega\rangle - p\sum_{i,j} Ric_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle \\ &\quad + (p-1)\sum_{i,j}(\overline{R}_\phi)_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle + \frac{1}{2}\sum_{i,j,k,l}\overline{R}_{ijkl}\langle i(e_j\wedge e_i)\omega, i(e_l\wedge e_k)\omega\rangle \\ &\quad + m(p-1)\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle \end{aligned}\tag{21}$$

From the hypotheses on the curvature of N and by techniques already used in the proof of the theorem 3.1 (see (14) and (15)) we deduce that

$$\begin{aligned}\frac{1}{2}|\mathcal{B}^-(\omega)|^2 &\leq \langle \mathcal{R}_p(\omega), \omega\rangle - p\sum_{i,j} Ric_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle \\ &\quad + p(p-1)\left((m-1)\overline{K}^1(x) + \overline{\rho}^1(x)\right)|\omega|^2 + m(p-1)\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle\end{aligned}\tag{22}$$

Now, let us estimate the last term. For this, assume that at the point $x \in M$, $(e_i)_{1\leq i\leq m}$ diagonalizes the symmetric tensor $\langle B(X,Y), H\rangle$. We have

$$\begin{aligned}\sum_{i,j}\langle B_{ij}, H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle &= \frac{1}{(p-1)!}\sum_{i,i_1,\dots,i_{p-1}}\langle B_{ii}, H\rangle\omega^2_{ii_1\dots i_{p-1}} \\ &= \frac{1}{p!}\sum_{i,i_1,\dots,i_{p-1}}\left(\langle B_{ii}, H\rangle + \langle B_{i_1i_1}, H\rangle + \dots + \langle B_{i_{p-1}i_{p-1}}, H\rangle\right)\omega^2_{ii_1\dots i_{p-1}} \\ &\leq \frac{1}{p!}\sum_{i,i_1,\dots,i_{p-1}}\left(|B_{ii}| + |B_{i_1i_1}| + \dots + |B_{i_{p-1}i_{p-1}}|\right)|H|\omega^2_{ii_1\dots i_{p-1}} \\ &\leq \frac{\sqrt{p}}{p!}\sum_{i,i_1,\dots,i_{p-1}}|B||H|\omega^2_{ii_1\dots i_{p-1}}\end{aligned}$$

Finally we have proved

$$\sum_{i,j} \langle B_{ij}, H\rangle \langle i(e_i)\omega, i(e_j)\omega\rangle \leq \sqrt{p}|B||H||\omega|^2 \tag{23}$$

Since $Ric \geq kg$, we have $\sum_{i,j} Ric_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle \geq pk|\omega|^2$, and we deduce from this, (22) and (23), the inequality of the proposition 4.1. □

The proof of the theorem 4.1 is now an immediate consequence of the proposition 4.1

PROOF OF THEOREM 4.1: Since $b_p(M) \neq 0$, there exists a nontrivial p-form ω so that $\Delta\omega = 0$. And from the Weitzenböck formula (5), we deduce

$$\int_M \langle \mathcal{R}_p(\omega), \omega\rangle \leq 0 \tag{24}$$

Now, applying the estimate of the proposition 4.1 we get the desired inequality. □

We can show a similar result to the theorem 4.1, with the scalar curvature $Scal$ of (M^m, g) instead of the Ricci curvature.

Theorem 4.2 *Let (M^m, g) be a compact Riemannian manifold of dimension $m \geq 2$ so that $b_p(M) \neq 0$ for a $p \geq 1$. Then for any isometric immersion ϕ from (M^m, g) into an n-dimensional Riemannian manifold (N^n, h), there exists at least a point $x \in M$ so that*

$$m\left(\frac{p-1}{\sqrt{p}} + \frac{m-p-1}{\sqrt{m-p}}\right)|B(x)||H(x)| \geq Scal(x) - (m-2)((m-1)\overline{K}^1 + \overline{\rho}^1)(\phi(x)) \tag{25}$$

We immediately deduce the following

Corollary 4.2 *Let (M^m, g) be a compact Riemannian manifold of dimension $m \geq 2$ minimally immersed into an n-dimensional Riemannian manifold (N^n, h) $(n > m)$. If $\overline{\rho}^1$ is bounded above and if*

$$\min_M(Scal) > (m-2)\left((m-1)\overline{K}^1_{max} + \overline{\rho}^1_{max}\right)$$

then for any $p \in \{1, ..., m\}$, we have $b_p(M) = 0$.

Remark 4.2: If $p = m/2$, we can improve (25) to obtain

$$m(m-2)|H(x)|^2 \geq Scal(x) - (m-2)((m-1)\overline{K}^1 + \overline{\rho}^1)(\phi(x)) \tag{26}$$

The theorem 4.2 and the inequality (26) was obtained by El Soufi (theorem 3.1 and theorem 3.2 of [5]) in the particular case where $m = 4$ and $p = 2$.

On the other hand, note that for $p \neq m/2$, the theorem 4.1 is not a consequence of the theorem 4.2.

For the same reasons as in the proof of theorem 3.2, we can choose locally an orientation on M and define locally the Hodge operator $\star$. But for all p-form ω of M, the quantity $\langle \mathcal{R}_{m-p}(\star\omega), \star\omega \rangle$ is globally defined.

The theorem 4.2 is a consequence of the following proposition

Proposition 4.2 *Let (M^m, g) be an m-dimensional compact Riemannian manifold isometrically immersed in an n-dimensional Riemannian manifold (N^n, h). Then for all $p \in \{1, ..., m-1\}$ and for all p-form ω on (M^m, g), we have for all $x \in M$*

$$\langle \mathcal{R}_p(\omega), \omega \rangle + \frac{p}{m-p} \langle \mathcal{R}_{m-p}(\star\omega), \star\omega \rangle \geq p \left(Scal(x) - (m-2) \left((m-1)\overline{K}^1 + \overline{\rho}^1 \right) (\phi(x)) - m \left(\frac{p-1}{\sqrt{p}} + \frac{m-p-1}{\sqrt{m-p}} \right) |H(x)||B(x)| \right) |\omega|^2$$

Remark 4.3: If $p = m/2$, we can improve this inequality (see the proof of the proposition 4.2) to obtain

$$\langle \mathcal{R}_p(\omega), \omega \rangle + \frac{p}{m-p} \langle \mathcal{R}_{m-p}(\star\omega), \star\omega \rangle \geq p \left(Scal(x) - (m-2) \left((m-1)\overline{K}^1 + \overline{\rho}^1 \right) (\phi(x)) - m(m-2)|H(x)|^2 \right) |\omega|^2 \quad (27)$$

And if for $p \geq 1$, $b_p(M) \neq 0$, we get (26).

Proof of the proposition 4.2: From the inequality (22) we obtain that for all $p \geq 1$

$$\begin{aligned} \langle \mathcal{R}_p(\omega), \omega \rangle &\geq p \sum_{i,j} Ric_{ij} \langle i(e_i)\omega, i(e_j)\omega \rangle \\ &- p(p-1) \left((m-1)\overline{K}^1 + \overline{\rho}^1 \right) (\phi(x)) |\omega|^2 \\ &- m(p-1) \sum_{i,j} \langle B_{ij}, H \rangle \langle i(e_i)\omega, i(e_j)\omega \rangle \end{aligned} \quad (28)$$

Since $\star\omega$ is a $(m-p)$-form we have also

$$\langle \mathcal{R}_{m-p}(\star\omega), \star\omega \rangle \geq (m-p) \sum_{i,j} Ric_{ij} \langle i(e_i) \star \omega, i(e_j) \star \omega \rangle - (m-p)(m-p-1) \left((m-1)\overline{K}^1 + \overline{\rho}^1 \right) (\phi(x)) |\omega|^2$$

$$- m(m-p-1)\sum_{i,j}\langle B_{ij},H\rangle\langle i(e_i)\star\omega, i(e_j)\star\omega\rangle \tag{29}$$

Multiplying (29) by $p/(m-p)$, and summing the obtained inequality with (28), we find

$$\begin{aligned}
\langle\mathcal{R}_p(\omega),\omega\rangle + \frac{p}{m-p}\langle\mathcal{R}_{m-p}(\star\omega),\star\omega\rangle \geq \\
& p\sum_{i,j}\left(Ric_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle + Ric_{ij}\langle i(e_i)\star\omega, i(e_j)\star\omega\rangle\right) \\
& - p(m-2)\left((m-1)\overline{K}^1 + \overline{\rho}^1\right)(\phi(x))|\omega|^2 \\
- m(p-1)\sum_{i,j}\langle B_{ij},H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle - mp\left(\frac{m-p-1}{m-p}\right)&\sum_{i,j}\langle B_{ij},H\rangle\langle i(e_i)\star\omega, i(e_j)\star\omega\rangle
\end{aligned}$$

By computations which are similar to (17) we get

$$\sum_{i,j}\left(Ric_{ij}\langle i(e_i)\omega, i(e_j)\omega\rangle + Ric_{ij}\langle i(e_i)\star\omega, i(e_j)\star\omega\rangle\right) = Scal|\omega|^2 \tag{30}$$

Thus

$$\begin{aligned}
\langle\mathcal{R}_p(\omega),\omega\rangle &+ \frac{p}{m-p}\langle\mathcal{R}_{m-p}(\star\omega),\star\omega\rangle \geq pScal|\omega|^2 \\
&- p(m-2)\left((m-1)\overline{K}^1 + \overline{\rho}^1\right)(\phi(x))|\omega|^2 \\
&- m(p-1)\sum_{i,j}\langle B_{ij},H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle \\
&- mp\left(\frac{m-p-1}{m-p}\right)\sum_{i,j}\langle B_{ij},H\rangle\langle i(e_i)\star\omega, i(e_j)\star\omega\rangle
\end{aligned} \tag{31}$$

$\Big($ Note that if $p = m/2$, then $p-1 = p\left(\frac{m-p-1}{m-p}\right)$, and we show with the same arguments as in the proof of (30) that

$$m(p-1)\sum_{i,j}\langle B_{ij},H\rangle\langle i(e_i)\omega, i(e_j)\omega\rangle + mp\left(\frac{m-p-1}{m-p}\right)\sum_{i,j}\langle B_{ij},H\rangle\langle i(e_i)\star\omega, i(e_j)\star\omega\rangle = p(m-2)m|H|^2|\omega|^2$$

and reporting this in (31), we obtain the inequality (27) of the remark 3.3. $\Big)$

From the estimate (23), we deduce

$$m(p-1)\sum_{i,j}\langle B_{ij},H\rangle\langle i(e_i)\omega,i(e_j)\omega\rangle\leq m(p-1)\sqrt{p}|H||B||\omega|^2$$

and

$$mp\left(\frac{m-p-1}{m-p}\right)\sum_{i,j}\langle B_{ij},H\rangle\langle i(e_i)\star\omega,i(e_j)\star\omega\rangle\leq\frac{mp(m-p-1)}{\sqrt{m-p}}|H||B||\omega|^2$$

now reporting these inequalities in (31), we find the inequality of the proposition. □

The proof of the theorem is now immediate

PROOF OF THEOREM 4.2: Let ω be a harmonic p-form. Since the Hodge operator commutes with the Laplacian, then $\star\omega$ is a harmonic $(m-p)$-form, and we have

$$\int_M\left(\langle\mathcal{R}_p(\omega),\omega\rangle+\frac{p}{m-p}\langle\mathcal{R}_{m-p}(\star\omega),\star\omega\rangle\right)\leq 0$$

and the theorem follows from the proposition 4.2. □

Remark 4.4: We can improve all the results of this section by considering the particular case of submanifolds of the complex projective space $\mathbb{C}P^n(c)$ $(c>0)$. We just need to compute in (21) the terms with the curvature tensor of the complex projective space. Then we obtain the same statements as previously by replacing $(m-1)\overline{K}^1+\overline{\rho}^1$ by $c/8((m-1)+3\parallel J_\phi\parallel^2$ where for all $x\in M$, $\parallel J_\phi\parallel(x)=\sup\{|J_\phi(X)|/X\in T_xM$ and $|X|=1\}$. In particular, if (M^m,g) is an m-dimensional compact Riemannian manifold minimally immersed in $\mathbb{C}P^n(c)$ and if

$$\min_M(Scal)>\frac{c(m-2)((m-1)+3\max_M(\parallel J_\phi\parallel^2))}{8}$$

then for any $p\in\{1,...,m\}$, we have $b_p(M)=0$.

Acknowledgements:

The author would like to express his gratefulness to Professor S.Ilias.

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